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Analyzing the Motion of Symmetric Tops Without Recurring to Analytical Mechanics

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Characterizing the dynamics of heavy symmetric tops is essential in several fields of theoretical and applied physics. Accordingly, a series of approaches have been developed to describe their motion. In this paper, we present a derivation based on elementary geometric considerations carried out in the laboratory frame. Our framework enabled the simple derivation of the equation of motion for small nutations. The introduced formalism is also employed to determine the alteration of the dynamics of heavy, symmetric, spinning tops in a rotating force field, that is compared to the precession characteristics of a quantum magnetic dipole in rotating magnetic field.

Keywords: spinning top, precession, geometric interpretation, small nutations, rotating force fields

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INTRODUCTION

Mainstream methods for determining the equation of motion of heavy symmetric tops can be classified according to the theoretical approaches used, and the reference frames applied. The framework can employ the toolkits of the more elementary Newtonian, or those of the analytical mechanics. The coordinate systems used include mixed ones (like certain triplets of Euler-angles), or rotating frames attached to the body (like the principal axes used when solving the Euler equations). Euler angles offer a natural parametrization of the rigid body attitude simply revealing the first integrals (constants of motion) within the framework of the Lagrangian formalism.

First, we recapitulate the most well known solutions developed up to this time. The majority of them [1–9] use the Euler-angles (φ precession angle, ψ spinning angle, ϑ nutation angle) to deduce the Euler-Lagrange equations.

The Euler angles φ and ψ (**Figure 1**) are cyclic coordinates with corresponding conserved conjugate momenta. These are the vector projections of the total angular momentum to the vertical axis, $L_z \equiv p_\varphi$, and onto the symmetry axis, namely $L_n \equiv p_\psi$. Finally, the Euler-Lagrange equation for the nutation angle, ϑ, is a second-order differential equation reducible to a first order equation by applying the conservation of the energy, E. The equation of motion for ϑ is uniquely determined by the three conserved quantities, L_n, L_z and E and is analogous to that of a particle in an effective potential. The time evolution of the other two angles, namely φ and ψ can be obtained as the direct integration of expressions in ϑ. This approach confers all three degrees of freedom distinct roles and different dynamics. Nutation, however, stands out of the triplet since it does not have an associated conserved quantity and unidirectionally modulates the other two degrees of freedom.

A series of methods to solve the problem of spinning tops avoid using analytical mechanics. Wittenburg [10] uses the Newton-Euler equation expressed in a precessing coordinate system. The textbook of Morin [11] also presents an elementary deduction, using Euler angles and a mixed system, where the Newton-Euler equation is also transformed to the precessing frame. In Ref. [12] it

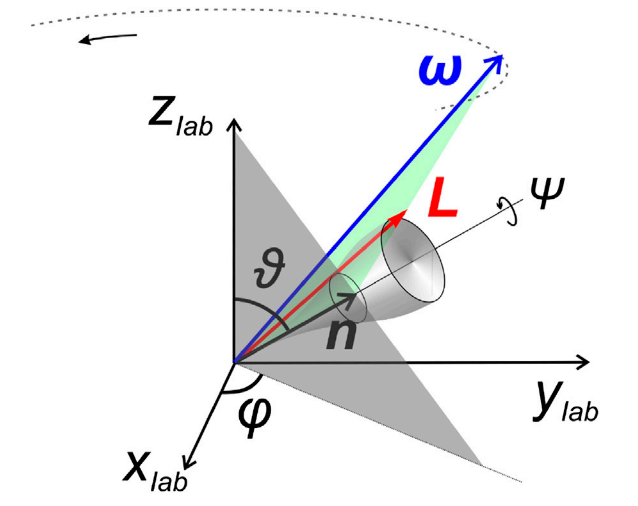

FIGURE 1 | Precessing and nutating heavy top. Angles φ (precession angle), ϑ (nutation angle) and ψ (spinning angle) parametrizing its attitude, the angular momentum, $\boldsymbol{L}$, and angular velocity, $\boldsymbol{\omega}$.

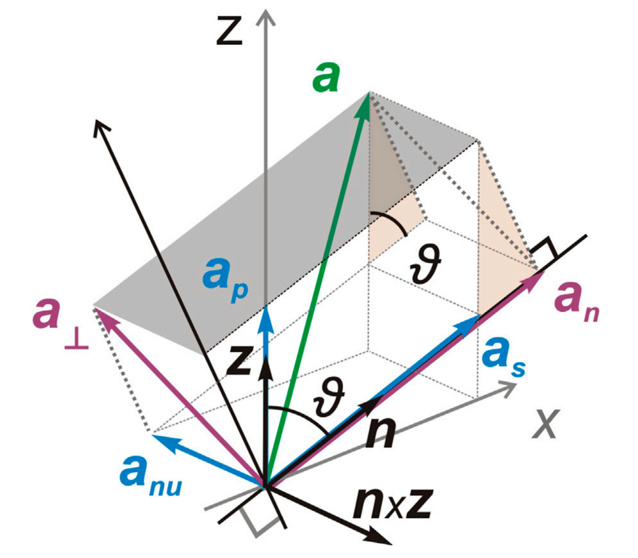

FIGURE 2 | Decomposition of a vector $\boldsymbol{a}$ in the non-orthogonal basis $\boldsymbol{n}$, $\boldsymbol{z}$, $\boldsymbol{n}\times\boldsymbol{z}$. For the definition of a_n, a_s and a_p and a_z, see **Eq. 1** and the corresponding text.

is shown that the three Euler-equations can be replaced by just as many conservation laws. Euler equations in rotating frame have also been applied to solve the problem [14].

As an alternative to the above more formal descriptions, pure precession has been intuitively explained by the so-called "square wheel model" where the spinning top is replaced by an ideal fluid flowing on a square-formed tube. This approach allows the explanation of the "hovering" of the top by forces acting on it, instead of the less intuitive conservation laws [15].

Here we present an alternative based on simple yet rigorous geometric considerations while employing only the elementary methods of Newtonian mechanics. The approach naturally leads to the separation of nutation from the other rotational degrees of freedom and makes possible the usage of a compact matrix formalism in the latter two dimensional subspace.

GEOMETRIC PRELIMINARIES

The spinning heavy top has two special directions that play an essential role in the relationships describing the dynamics of its vectorial quantities. One is the symmetry axis $\boldsymbol{n}$, while the other one is the direction of the gravitational field $\boldsymbol{z}$ (see **Figure 1**). These two unit vectors, spanning a plane, and the direction orthogonal to this plane, namely $\boldsymbol{e}_{\rm nu} \equiv \boldsymbol{n}\times\boldsymbol{z}/|\boldsymbol{n}\times\boldsymbol{z}|$, serve as a natural basis for investigating our three dimensional model. The spontaneous emergence of this basis is the reason behind the incontestable usefulness of Euler angles ψ, φ and ϑ for specifying the orientation of a spinning, symmetric rigid body. The rates of change of the these angles are denoted by $\dot\psi \equiv \omega_s, \dot\varphi \equiv \omega_p$ and $\dot\vartheta \equiv \omega_{\rm nu}$. Using the above basis, any vector $\boldsymbol{a}$ can be decomposed as

$$\boldsymbol{a} = a_s\boldsymbol{n} + a_p\boldsymbol{z} + a_{\rm nu}\boldsymbol{e}_{\rm nu} = \boldsymbol{a}_s + \boldsymbol{a}_p + \boldsymbol{a}_{\rm nu}, \tag{1}$$

where the three terms are vector projections of $\boldsymbol{a}$ parallel to the respective basis vectors (see **Figure 2**). Since the chosen basis is not orthogonal, the scalar projections $a_n = \boldsymbol{a}\cdot\boldsymbol{n}$, $a_z = \boldsymbol{a}\cdot\boldsymbol{z}$ and $a_{\rm nu} = \boldsymbol{a}\cdot\boldsymbol{e}_{\rm nu}$ also claim a role in the description. Alternatively, one can project $\boldsymbol{a}$ to one of the basis vectors and to the corresponding orthogonal plane:

$$\begin{aligned} \boldsymbol{a} &= \boldsymbol{a}_n + \boldsymbol{a}_\perp, \\ \boldsymbol{a}_n &= \boldsymbol{n}a_n = (\boldsymbol{n}\circ\boldsymbol{n})\boldsymbol{a}, \\ \boldsymbol{a}_\perp &= (\boldsymbol{a}-\boldsymbol{a}_n) = (\mathbb{I}-\boldsymbol{n}\circ\boldsymbol{n})\boldsymbol{a}. \end{aligned} \tag{2}$$

The dynamics of the top is such that these three directions are associated with qualitatively different phenomena (spin, precession and nutation). The nutation stands out of the trio as will become apparent also from this study. Therefore we shall introduce a formalism that manifestly separates the description into aspects confined to the rotating $(\boldsymbol{n},\boldsymbol{z})$ plane and aspects involving the direction perpendicular to it. Due to the linear connection between different decompositions and between kinematic and dynamic quantities such as angular velocity and angular momentum a matrix formalism will be useful.

Figure 2 reveals a number of geometric relations including

$$\begin{aligned} a_n &= a_s + \cos\vartheta a_p, \\ a_z &= a_p + \cos\vartheta a_s, \end{aligned} \tag{3}$$

that can be expressed compactly as

$$\boldsymbol{a}_{n,z} = \widehat{\boldsymbol{G}}\boldsymbol{a}_{s,p}, \tag{4}$$

where

$$\boldsymbol{a}_{n,z} = \begin{pmatrix} a_n \\ a_z \end{pmatrix}, \qquad \boldsymbol{a}_{s,p} = \begin{pmatrix} a_s \\ a_p \end{pmatrix},$$

$$\hat{\boldsymbol{G}} \equiv \begin{pmatrix} 1 & u \\ u & 1 \end{pmatrix}, \qquad \hat{\boldsymbol{G}}^{-1} = \frac{1}{s^2}\begin{pmatrix} 1 & -u \\ -u & 1 \end{pmatrix},$$

with

$$u = \cos\vartheta, \qquad s = \sqrt{1-u^2} = \sin\vartheta,$$

and

$$(\boldsymbol{a}_z - \boldsymbol{a}_n)^2 = \sin^2\vartheta\left(\boldsymbol{a}_s + \boldsymbol{a}_p\right)^2, \tag{5}$$

that will be applied *in Time Evolution of the Spin and Precession Angles*. The proof for **Eq. 5** is shown in *Proof of* ***Eq. (5)*** of the **Supplementary Material**. Note that the connections between a_n, a_z, a_s and a_p are solely determined by ϑ.

RELATIONSHIPS BETWEEN THE COMPONENTS OF L AND ω

The components of the angular momentum along the symmetry axis $\boldsymbol{n}$ and the orthogonal ones to this are referred to as $L_n = C\omega_n$, $L_\perp = A\omega_\perp$, where C and A are the corresponding principal moments of inertia.

Therefore,

$$\begin{aligned} \boldsymbol{L} = \boldsymbol{L}_n + \boldsymbol{L}_\perp &= [C\boldsymbol{n}\circ\boldsymbol{n} + A(\mathbb{I} - \boldsymbol{n}\circ\boldsymbol{n})]\boldsymbol{\omega} \\ &= A\boldsymbol{\omega} + (C-A)\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{\omega}) = A\boldsymbol{\omega} + (C-A)\omega_n\boldsymbol{n}. \end{aligned} \tag{6}$$

The above linear interdependence between $\boldsymbol{L}$, $\boldsymbol{\omega}$ and $\boldsymbol{n}$ reveals their coplanarity. Note that for asymmetric tops this property does not hold.

Using the notation introduced in *Geometric Preliminaries*, **Eq. 6** can be rewritten as

$$\boldsymbol{L}_{n,z} = C\hat{\boldsymbol{D}}\boldsymbol{\omega}_{n,z},\ L_{\mathrm{nu}} = A\omega_{\mathrm{nu}}, \tag{7}$$

where

$$\hat{\boldsymbol{D}} \equiv \begin{pmatrix} 1 & 0 \\ (1-\alpha)u & \alpha \end{pmatrix}, \quad \alpha = A/C.$$

READING CONSERVED QUANTITIES L_n, ω_n, AND L_z

Let us to consider the Newton-Euler equation

$$\dot{\boldsymbol{L}} = w\boldsymbol{z}\times\boldsymbol{n}, \tag{8}$$

where w is the magnitude of the torque of the homogeneous gravitational field pointing into the $-\boldsymbol{z}$ direction. Due to **Eq. 8** we have $\boldsymbol{n}\cdot\dot{\boldsymbol{L}} = 0$. All points of the top are engaged in a rotation defined by $\boldsymbol{\omega}$. This is also true for the symmetry axis $\boldsymbol{n}$, that is,

$$\dot{\boldsymbol{n}} = \boldsymbol{\omega}\times\boldsymbol{n}, \tag{9}$$

revealing that $\dot{\boldsymbol{n}}$ is orthogonal to the plane spanned by $\boldsymbol{\omega}$ and $\boldsymbol{n}$. Due to the co-planarity of $\boldsymbol{L}$, $\boldsymbol{\omega}$ *and* $\mathbf{n}$ *we have* $\dot{\boldsymbol{n}}\cdot\boldsymbol{L} = 0$. *Therefore*

$$\dot{L}_n = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{L}\cdot\boldsymbol{n}) = 0,$$

thus L_n is conserved. **Equation 7** entails that ω_n is conserved as well.

A similar but more straightforward consideration yields $L_z = \boldsymbol{z}\cdot\boldsymbol{L} =$ const. as $\dot{\boldsymbol{z}} = 0$. Note that since no dissipation is present, the energy of the system is also conserved.

TIME EVOLUTION OF THE SPIN AND PRECESSION ANGLES

Due to the conservation of the angular momentum components L_n and L_z it is worth connecting them directly with the kinematically relevant spin and precession angular velocities. Combining **Eqs 4** and **7** results in

$$\boldsymbol{L}_{n,z} = C\hat{\boldsymbol{T}}\boldsymbol{\omega}_{s,p}, \qquad \hat{\boldsymbol{T}} = \hat{\boldsymbol{D}}\hat{\boldsymbol{G}} = \begin{pmatrix} 1 & u \\ u & \alpha s^2 + u^2 \end{pmatrix}.$$

This enables the expression of the two angular velocities as

$$\boldsymbol{\omega}_{s,p} = \frac{1}{C}\hat{\boldsymbol{T}}^{-1}\boldsymbol{L}_{n,z}, \qquad \hat{\boldsymbol{T}}^{-1} = \frac{1}{\alpha s^2}\begin{pmatrix} \alpha s^2 + u^2 & -u \\ -u & 1 \end{pmatrix}. \tag{10}$$

This formula has a pivotal importance: it connects the kinematic quantities of interest to the conserved dynamic quantities.

Note that ω_s and ω_p solely depend on conserved components of angular momenta and the time-dependent polar angle $\vartheta(t)$ therefore become themselves constants of motion if $\omega_{\mathrm{nu}} \equiv \dot{\vartheta} = 0$, phenomenon called pure precession.

TIME EVOLUTION OF THE NUTATION ANGLE

The above results were obtained without making explicit reference to the conservation of energy. For moving beyond pure precession and describing nutation we have to quantify the migration of the energy between kinetic and potential components during nutation.

By expressing the angular frequency $\boldsymbol{\omega}$ from the linear **Eq. 6** the rotational energy of the top can be written as

$$T \equiv \frac{1}{2}\boldsymbol{L}\cdot\boldsymbol{\omega} = \frac{1}{2A}\boldsymbol{L}^2 + \frac{1}{2}\left(1 - \frac{C}{A}\right)\omega_n L_n, \tag{11}$$

while the potential energy reads

$$V(\vartheta) = w\cos\vartheta. \tag{12}$$

Exploiting the orthogonality of $\boldsymbol{e}_{\mathrm{nu}}$ to the $(\boldsymbol{n}, \boldsymbol{z})$ plane combined with property **Eq. 5** we get

$$\boldsymbol{L}^2 = \left(\boldsymbol{L}_s + \boldsymbol{L}_p\right)^2 + \boldsymbol{L}_{\mathrm{nu}}^2 = \frac{(\boldsymbol{L}_n - \boldsymbol{L}_z)^2}{\sin^2\vartheta} + \boldsymbol{L}_{\mathrm{nu}}^2. \tag{13}$$

The dynamics ruling the nutation angle can be regarded as a one-dimensional motion in an effective potential, motion completely determined by the conservation of the effective energy. Having L^2 and V obtained, enables us to provide the formula for these effective energies

$$E_{\text{eff}} = T_{\text{eff}}\left(\dot{\vartheta}\right) + V_{\text{eff}}(\vartheta), \tag{14}$$

where the reuse of **Eqs 7**, **11** and **13** gives

$$T_{\text{eff}}\left(\dot{\vartheta}\right) = \frac{1}{2A}\boldsymbol{L}_{\text{nu}}^2 = \frac{A}{2}\dot{\vartheta}^2,$$
$$V_{\text{eff}}(\vartheta) = \frac{1}{2A}\frac{(\boldsymbol{L}_n - \boldsymbol{L}_z)^2}{\sin^2\vartheta} + V(\vartheta),$$
$$E_{\text{eff}} = E - \frac{1}{2}\left(\frac{1}{C} - \frac{1}{A}\right)L_n^2.$$

For convenience **Eq. 14** can be rewritten in terms of $u \equiv \cos\vartheta$ and $\dot{u} = -\sqrt{1-u^2}\dot{\vartheta}$ as

$$\frac{A}{2}\dot{u}^2 + U_{\text{nu}}(u) = \epsilon, \tag{15}$$

where

$$U_{\text{nu}}(u) = \nu u + \frac{\kappa}{2}u^2 - \gamma u^3.$$

Here the Greek letters denote the following

$$\nu = w - \frac{L_n L_z}{A}, \tag{16}$$
$$\kappa = 2E - L_n^2\left(\frac{1}{C} - \frac{1}{A}\right),$$
$$\gamma = w,$$
$$\epsilon = E - \frac{L_n^2}{2C} - \frac{L_z^2}{2A}.$$

The above relationships reveal that the equation of motion for the nutation angle, ϑ, can be solved decoupled from the other two angles, namely the φ precession and ψ spinning angle.

Full solution of the problem requires to resolve the time evolution of Euler angles. **Equation 15** rules $\vartheta(t)$, while $\psi(t)$ and $\varphi(t)$ can be determined by integrating ω_s, repectively ω_p in **Eq. 10**.

SMALL NUTATIONS

In order to describe small nutations, we consider the minimum point u_0 of the one-dimensional potential $U_{\text{nu}}(u)$:

$$\frac{\mathrm{d}U_{\text{nu}}}{\mathrm{d}u}(u_0) = \nu + \kappa u_0 - 3\gamma u_0^2 = 0, \tag{17}$$
$$\frac{\mathrm{d}^2U_{\text{nu}}}{\mathrm{d}u^2}(u_0) = \kappa - 6\gamma u_0 = A\Omega_{\text{nu}}^2 > 0\,.$$

providing

$$u_0 = \frac{\kappa - \sqrt{\kappa^2 + 12\nu\gamma}}{6\gamma},$$
$$\Omega_{\text{nu}} = \sqrt{\frac{\kappa^2 + 12\nu\gamma}{A}},$$

where Ω_{nu} represents the angular frequency of small, nearly harmonic oscillations in the polar angle during nutation.

We intend to investigate small deviations from pure precession. In the presence of nutation, the conservation of quantities such as ω_s, ω_p or $\boldsymbol{L}^2$ does not hold any more.

In the low amplitude oscillation limit, $\Delta u(t) = u(t) - u_0 = \delta\cdot\cos(\Omega_{\text{nu}}t)$, $\delta \ll 1$ and $\dot{u}(t) = -\Omega_{\text{nu}}\delta\sin(\Omega_{\text{nu}}t) = -\Omega_{\text{nu}}\sqrt{\delta^2 - [\Delta u(t)]^2}$ is an oscillation with the same frequency but $\pi/2$ phase delay.

Let us to denote generically by $f(u)$ the physical quantities modulated by the nutation angle. Since the temporal alteration of f can be written as $\Delta f[u(t)] \approx f'(u_0)\Delta u(t)$, the physical quantities modulated by u will oscillate with the same frequency. Moreover, f will oscillate with an amplitude $f'(u_0)\delta$ around the mean value, $f(u_0)$, that is the pure precession value at u_0.

The deviation of the angular frequency components can be obtained from **Eq. 10**

$$\Delta\boldsymbol{\omega}_{s,p} = \Delta u \frac{d\widehat{\boldsymbol{T}}^{-1}}{du}\bigg|_{u_0}\frac{\boldsymbol{L}_{n,z}}{C},$$

where we made use of the conserved character of $\boldsymbol{L}_{n,z}$ and assume the time dependence of u as implicit. By definition the nutation component of the angular frequency is

$$\omega_{\text{nu}} = \dot{\theta} = -\frac{\dot{u}}{\sqrt{1-u^2}}.$$

For any vectorial quantity, $\boldsymbol{A}$, with rate of change $\dot{\boldsymbol{A}}$ its nutation motion, $\Delta\dot{\boldsymbol{A}}$, can be described as that of a time dependent geometric vector viewed from the purely precessing reference frame rotating with $\omega_p(u_0)\boldsymbol{z}$. The transformation to the rotating (precessing) reference frame is given by

$$\Delta\dot{\boldsymbol{A}} = \dot{\boldsymbol{A}} - \omega_p(u_0)\boldsymbol{z}\times\boldsymbol{A}. \tag{18}$$

The nutation of the symmetry axis, $\dot{\boldsymbol{n}}$, can be captured by combining **Eq. 18** with **Eqs 1** and **9**. In the laboratory frame its rate of change can be written as

$$\dot{\boldsymbol{n}} = \left(\omega_p\boldsymbol{z} + \omega_{\text{nu}}\boldsymbol{e}_{\text{nu}}\right)\times\boldsymbol{n}.$$

According to **Eq. 18**

$$\dot{\boldsymbol{n}}_{\text{nu}} = \left(\Delta\omega_p\boldsymbol{z} + \omega_{\text{nu}}\boldsymbol{e}_{\text{nu}}\right)\times\boldsymbol{n} = -s\Delta\omega_p\boldsymbol{e}_{\text{nu}} + \omega_{\text{nu}}\boldsymbol{e}_{\perp}.$$

The two orthogonal terms are proportional to Δu and $\dot{u}$, respectively, indicating a rotational movement about the (purely) precessing symmetry axis. From **Eqs 11** and **12** we can see that

$$\frac{1}{2A}\boldsymbol{L}^2 = E - \frac{1}{2}\left(1 - \frac{C}{A}\right)\omega_n L_n + wu. \tag{19}$$

Apart from u all other quantities are either parameters or constants of motion revealing that during small nutations the square of the total angular momentum oscillates with amplitude $2AW\delta$ and frequency Ω_{nu} about its pure precession value.

PURE PRECESSION

By setting $\dot{u} = 0$ **Eqs 15** and **17** take the form

$$\nu u + \frac{\kappa}{2} u^2 - \gamma u^3 = \epsilon,$$
$$\nu + \kappa u - 3\gamma u^2 = 0,$$

yielding

$$\kappa = 3\gamma u - \frac{\nu}{u}, \qquad 2\epsilon = u\left(\nu + \gamma u^2\right). \tag{20}$$

By eliminating the energy, E, from the definitions of κ and ϵ in **Eq. 16**, we have

$$\kappa - 2\epsilon = \frac{L_n^2 + L_z^2}{A}.$$

Combining the above with **Eq. 20** we get

$$L_n^2 + L_z^2 - 2L_n L_z \chi = \boldsymbol{L}_{n,z}^\top \widehat{\boldsymbol{Q}} \boldsymbol{L}_{n,z} = -\frac{\alpha s^4}{u} C w,$$

where $\chi = \frac{1}{2}\left(u + \frac{1}{u}\right)$, $s^2 = 1 - u^2$, and

$$\widehat{\boldsymbol{Q}} = \begin{pmatrix} 1 & -\chi \\ -\chi & 1 \end{pmatrix}.$$

Therefore

$$\boldsymbol{\omega}_{s,p}^\top \widehat{\boldsymbol{T}}^\top \widehat{\boldsymbol{Q}} \widehat{\boldsymbol{T}} \boldsymbol{\omega}_{s,p} = -\frac{\alpha s^4}{u} \frac{w}{C},$$

wherein

$$\widehat{\boldsymbol{T}}^\top \widehat{\boldsymbol{Q}} \widehat{\boldsymbol{T}} = \frac{\alpha s^4}{u} \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & u(\alpha - 1) \end{pmatrix},$$

resulting in

$$\omega_p^2 \cos\vartheta (A - C) - \omega_p \omega_s C + w = 0,$$

the well-known relationship between precession and spin angular velocities for a given value of the nutation angle ϑ.

PURE PRECESSION IN A ROTATING FORCE FIELD

Spins driven by rotating magnetic fields have been extensively studied due to their importance in resonance spectroscopy. Here we will study the effect of a horizontally rotating homogeneous field on a classical gyroscope. This force can be implemented, for example, by electrostatic interactions. In this case, the motion of a heavy spinning top without dissipation generally becomes erratic. Therefore we will limit our investigation to the situation when the precession is in synchrony with the driving field, meaning, that the rotating component of the field stays in the same vertical plane as the symmetry axis. In these special circumstances, the equations connecting kinematic and dynamic quantities such as **Eqs 10** and **11** are not affected by the particularities of the field. However, the conservation laws derived in Reading Conserved *Quantities* L_n, *!n, and* L_Z depend on the geometric relationship between the field and the symmetry axis of the top. If kept in the $(\boldsymbol{n}, \boldsymbol{z})$ plane the rotating field component will only change the magnitude of the torque in **Eq. 8** and not its direction. However, the potential energy in **Eq. 12** will modify as

$$\tilde{V}(\vartheta) = V(\vartheta) + b \sin\vartheta, \tag{21}$$

where b quantifies the effect of the horizontally rotating field component leading to the one dimensional effective potential

$$\tilde{U}_{\mathrm{nu}}(u) = U_{\mathrm{nu}}(u) + b\left(1 - u^2\right)^{3/2}. \tag{22}$$

Since the exhaustive investigation of the properties of the above function is beyond the scope of this paper we only remark that the main features of the dynamics are not affected by the additional term from above. For a simple yet quantitative conclusion we further confine our study to the limit of weak driving fields and view $\tilde{U}_{\mathrm{nu}}$ as the perturbation of U_{nu}. The stable solution of the perturbed nutation angle $\tilde{u}_0$ can be obtained from

$$\frac{\mathrm{d}\tilde{U}_{\mathrm{nu}}}{\mathrm{d}u}(\tilde{u}_0) = \frac{\mathrm{d}U_{\mathrm{nu}}}{\mathrm{d}u}(\tilde{u}_0) - 3b\tilde{u}_0\sqrt{1 - \tilde{u}_0^2} = 0. \tag{23}$$

The first order Taylor expansion around u_0 gives

$$\tilde{u}_0 - u_0 = b\frac{u_0\sqrt{1 - u_0^2}}{A\Omega_{\mathrm{nu}}^2}. \tag{24}$$

The above expansion procedure applied on the second derivative of $\tilde{U}_{\mathrm{nu}}$ yields

$$\frac{\mathrm{d}^2\tilde{U}_{\mathrm{nu}}}{\mathrm{d}u^2}(\tilde{u}_0) - \frac{\mathrm{d}^2 U_{\mathrm{nu}}}{\mathrm{d}u^2}(u_0) = \mathcal{O}(b), \tag{25}$$

ensuring the stability of the perturbed solution. Note that the conclusions on the existence and stability of the stationary solution can be extended well beyond the perturbative range of the driving field component.

PRECESSING SPIN IN A ROTATING MAGNETIC FIELD

In a broader context precession is a term applicable to any axis with one of its points fixed and performing a circular motion along the surface of a cone. Outside the realm of inertial macroscopic motion [16] we encounter it in quantum mechanics of magnetic dipoles and it is the basis of nuclear magnetic resonance (NMR) [17] and ESR [18]. Atomic systems in strong fields obey dynamics where inertia has little or no role. However, the nature of the coupling between the angular momentum and the magnetic field produces a motion that is similar to the precession of a rigid body.

Let us consider a magnetic field that has a constant vertical and a rotating horizontal component, namely $\boldsymbol{B} = [b\sin(\omega t), b\cos(\omega t), B]$. Note that the horizontal component here rotates counterclockwise with respect to the third axis. The equation of motion for the quantum mechanical expectation value, $\boldsymbol{S}$, of the angular momentum coupled through the gyromagnetic factor γ to this field reads

$$\dot{\boldsymbol{S}} = \gamma \boldsymbol{S} \times \boldsymbol{B}. \tag{26}$$

Note that if no horizontal rotating component is present $\boldsymbol{S}$ precesses with the Larmor frequency $\omega_L = \gamma B$ (See ***Spin in Magnetic Field*** in the **Supplementary Material**). In this special case the attitude of $\boldsymbol{S}$ is arbitrary, i.e., determined by the initial condition. In the presence of dissipation the angle will relax to zero, i.e., parallel to the constant magnetic field.

In the general case, when the rotating component of the magnetic field is present, the stationary (particular) solution of **Eq. 26** will be a precession motion with the same ω frequency as the driving field and the angle φ enclosed with the vertical is

$$\cot\varphi = \frac{\omega_L - \omega}{\omega_l}, \quad \omega_l \equiv \gamma b. \tag{27}$$

Note that transients are disregarded. During this deduction the laboratory reference frame was used.

Though both refer to angular momenta, **Eqs 8** and **26** are far from being equivalent. The cross product in **Eq. 26** conserves the magnitude of the angular momentum. Therefore the magnitude oscillations described by **Eq. 19** are not present in the case of spins.

DISCUSSION

The dynamics of a heavy symmetric top is determined by the constants of motion L_n, L_z and E. An essential output of our approach is expressed in **Eq. 10**. This relationships represents the inversion in the $(\boldsymbol{n}, \boldsymbol{z})$ subspace of the linear **Eq. 6** such that the angular velocities are expressed in terms of L_n and L_z. The momentum $p_\vartheta = L_{\text{nu}}$ associated with the third coordinate, ϑ, is not conserved. Nutation "remains alone" in a first order differential equation describing a one dimensional non-harmonic oscillator (see **Eq. 15**). This periodic conversion of the energy from potential to kinetic and back will modulate the spin and precession angular velocities through **Eq. 10**.

In the case of small nutations the only effective geometric parameter is the nutation angle ϑ characterizing the attitude of the top. The magnitude of the angular momentum harmonically oscillates around its value encountered in pure precession.

We also examine the case of the classical symmetric spinning rigid body and the quantum mechanical spin (without inertia) precessing in an external field having a rotating component. While the main features of the spin dynamics can be provided analytically, the case of a heavy spinning top driven by a rotating field seems to be more complex. For the case without dissipation, the dynamics of the system will be unpredictable, except the case when the precession frequency is in synchrony with the driving - the case discussed in first-order approximation in this paper.

Our paper employing matrix formalism combined with geometry provides another example that the problem of spinning top can be addressed by a multitude of approaches, each emphasizing a different facet of the phenomenon.

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/**Supplementary Material**, further inquiries can be directed to the corresponding author.

AUTHOR CONTRIBUTIONS

All authors listed have made a substantial, direct, and intellectual contribution to the work and approved it for publication.

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SUPPLEMENTARY MATERIAL

The Supplementary Material for this article can be found online at: https://www.frontiersin.org/articles/10.3389/fphy.2020.584294/full#supplementary-material.

REFERENCES

1. Johns O. *Analytical mechanics for relativity and quantum mechanics.* New York, NY: Oxford University Press (2005) p. 656.
2. Lemos NA. *Analytical mechanics.* Cambridge, UK: Cambridge University Press (2018) p. 459.
3. Hand LN, Finch JD. *Analytical mechanics.* Cambridge, UK: Cambridge University Press (1998) p. 575.
4. Landau LD, Lifshitz EM. Mechanics. *Course of theoretical physics.*, Vol. 1. Oxford, UK: Pergamon Press (1969) p. 197.
5. Herbert Goldstein CP, Safko J. *Classical mechanics.* 3rd ed. London, UK: Pearson (2001) p. 20.
6. Lüdde CS, Dreizler RM. *Theoretical mechanics.* Berlin, Germany: Springer (2010) p. 402.
7. Ginsberg JH. *Engineering dynamics.* Cambridge, UK: Cambridge University Press (2008) p. 742.
8. Kibble TWB, Berkshire FH. *Classical mechanics.* London, UK: Imperial College Press (2004) p. 478.
9. Gregory RD. *Classical mechanics.* Cambridge, UK: Cambridge University Press (2006) p. 596.
10. Wittenburg J. *Dynamics of multibody systems.* Berlin, Germany: Springer (2008) p 223.
11. Morin D. *Introduction to classical mechanics.* Cambridge, UK: Cambridge University Press (2008) p. 719.
12. Romano A, Marasco A. *Classical mechanics with Mathematica®.* Cham, Switzerland: Springer International Publishing (2018) p. 506.
13. Berry MV, Shukla P. Slow manifold and hannay angle in the spinning top. *Eur J Phys.* (2010) 32:115–27. doi:10.1088/0143-0807/32/1/011

14. Bhattacharjee S. Rotating frame analysis of rigid body dynamics in space phasor variables. *Am J Phys.* (2013) 81:518–26. doi:10.1119/1.4803531
15. Hantz P, Lázár ZI. Precession intuitively explained. *Front Phys.* (2019) 7:5. doi:10.3389/fphy.2019.00005
16. Itu C, Öchsner A, Vlase S, Marin MI, Improved rigidity of composite circular plates through radial ribs. *Proc Inst Mech Eng L J Mater.* (2018) 233:1585–93. doi:10.1177/1464420718768049
17. Williamson MP. Drawing single NMR spins and understanding relaxation. *Nat Prod Commun.* (2019) 14:1–9. doi:10.1177/1934578x19849790
18. Bertrand P. *Electron paramagnetic resonance spectroscopy: fundamentals.* Cham, Switzerland: Springer (2020) p. 420.

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